

Asymptotics for the Wiener Sausage among Poissonian Obstacles

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Received: 11 September 2007 / Accepted: 1 October 2008 / Published online: 18 October 2008
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Abstract We consider the Wiener sausage among Poissonian obstacles. The obstacle is called *hard* if Brownian motion entering the obstacle is immediately killed, and is called *soft* if it is killed at certain rate. It is known that Brownian motion conditioned to survive among obstacles is confined in a ball near its starting point. We show the weak law of large numbers, large deviation principle in special cases and the moment asymptotics for the volume of the corresponding Wiener sausage. One of the consequence of our results is that the trajectory of Brownian motion almost fills the confinement ball.

Keywords Brownian motion · Poissonian obstacles · Wiener sausage

1 Introduction

We consider Brownian motion conditioned to avoid Poissonian obstacles. It is known that conditional Brownian motion typically localizes in a ball near its starting point under the annealed measure. In this article, we show that the trajectory of the particle almost fills the ball in which it is confined.

We shall start by introducing the notation and the model. Let Ω be the set of locally finite simple pure point measures on \mathbb{R}^d and \mathbb{P}_ν be the Poisson point process of constant intensity ν on Ω . For a fixed nonpolar compact subset K of \mathbb{R}^d and $\Omega \ni \omega = \sum_i \delta_{x_i}$, we define the hard obstacles $S(\omega) = \bigcup_i (x_i + K)$. Similarly, for a nonnegative, compactly supported and bounded measurable function W which is not identically zero and $\Omega \ni \omega = \sum_i \delta_{x_i}$, we define the soft obstacles $V(x, \omega) = \sum_i W(x - x_i)$. Next we introduce Brownian motion Z on \mathbb{R}^d independent of the Poisson point process. The law of Z conditioned to start from $x \in \mathbb{R}^d$ is denoted by P_x and E_x stands for the corresponding expectation. For an open set $U \subset \mathbb{R}^d$ and a closed set $F \subset \mathbb{R}^d$, $T_U = \inf \{s \geq 0; Z_s \notin U\}$ and $H_F = \inf \{s \geq 0; Z_s \in F\}$ are the exit time of U and the entrance time of F , respectively.

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We define the annealed path measure

$$Q_t^{\mu, \nu} = \frac{1}{S_t^{\mu, \nu}} \exp \left\{ - \int_0^t V(Z_s, \omega_1) ds \right\} 1_{\{H_{S(\omega_2)} > t\}} \mathbb{P}_\mu^1 \otimes \mathbb{P}_\nu^2 \otimes P_0$$

on $\Omega^2 \times C([0, t], \mathbb{R}^d)$ with $S_t^{\mu, \nu}$ the normalizing constant:

$$S_t^{\mu, \nu} = \mathbb{E}_\mu^1 \otimes \mathbb{E}_\nu^2 \otimes E_0 \left[\exp \left\{ - \int_0^t V(Z_s, \omega_1) ds \right\}; H_{S(\omega_2)} > t \right]$$

with obvious notations. This path measure describes the behavior of Brownian motion among the killing traps conditioned not to be killed up to time t . The first mathematical result concerned with this measure is Donsker-Varadhan’s work [3] about asymptotics for $S_t^{\nu, 0}$ with W as before and $S_t^{0, \nu}$ in the case K is a closed ball of arbitrary fixed radius. They showed, using large deviation technique, that

$$S_t^{0, \nu} \text{ (or } S_t^{\nu, 0}) = \exp \left\{ -c(d, \nu) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \quad (t \rightarrow \infty)$$

where

$$c(d, \nu) = \inf_{U \subset \mathbb{R}^d: \text{open}} \{ \nu|U| + \lambda(U) \} \tag{1}$$

with $|U|$ the Lebesgue measure of U and $\lambda(U)$ the principal Dirichlet eigenvalue of $-1/2\Delta$ in U . It follows from Faber-Krahn’s inequality (see e.g. [1]) that balls with radius

$$R_0(d, \nu) = \left(\frac{2\lambda_d}{d\nu\omega_d} \right)^{\frac{1}{d+2}} \tag{2}$$

achieve the infimum in (1). Here ω_d is the volume of the d -dimensional unit ball $B(0, 1)$ and λ_d the principal Dirichlet eigenvalue of $-1/2\Delta$ in $B(0, 1)$. Therefore we can obtain the exact value of $c(d, \nu)$:

$$c(d, \nu) = \frac{d+2}{2} (\nu\omega_d)^{\frac{2}{d+2}} \left(\frac{2\lambda_d}{d} \right)^{\frac{d}{d+2}}.$$

Sznitman generalized this result to $S_t^{0, \nu}$ with arbitrary nonpolar compact K and also improved the asymptotic estimates as follows:

$$\begin{aligned} \exp \left\{ -c(d, \nu) t^{\frac{d}{d+2}} - \gamma(a, d, \nu) t^{\frac{d-1}{d+2}} \right\} &\leq S_t^{0, \nu} \text{ (or } S_t^{\nu, 0}) \\ &\leq \exp \left\{ -c(d, \nu) t^{\frac{d}{d+2}} + t^{\frac{d\mu(d)}{d+2}} \right\} \end{aligned} \tag{3}$$

for large t , using his ‘method of enlargement of obstacles’ (see Theorem 4.5.6 in [7]). Here $\gamma(a, d, \nu) > 0$ and $\mu(d) \in (0, 1)$ are constants and a is defined via

$$a = \inf \{ u > 0; K \text{ (resp. } \text{supp}(W)) \subset \bar{B}(0, u) \}. \tag{4}$$

Sznitman ($d = 2$, in [5]) and Povel ($d \geq 3$, in [4]), motivated by the proof of the lower bound in (3), showed that surviving Brownian particle is typically confined in a ball with radius $t^{1/(d+2)} R_0$ for large t .

Theorem 1 (Confinement property) *Let $d \geq 2$. There exist constants $\kappa_1 > 1$ and $0 < \kappa_2 < 1$ and for each $(\omega_1, \omega_2) \in \Omega^2$ a ball $B(\omega_1, \omega_2)$ with center in $B(0, R_0(d, \mu + \nu) + \kappa_1 t^{-\kappa_2/(d+2)})$ and with radius in $[R_0(d, \mu + \nu), R_0(d, \mu + \nu) + \kappa_1 t^{-\kappa_2/(d+2)}]$ such that*

$$\lim_{t \rightarrow \infty} Q_t^{\mu, \nu}(T_t^{1/(d+2)} B(\omega_1, \omega_2) > t) = 1.$$

Although Sznitman and Povel showed this theorem only in the case $Q_t^{0, \nu}$, their argument is easily applicable to above version. As a consequence of this property, the volume of the Wiener sausage $W_t^C = \bigcup_{s \leq t} (Z_s + C)$ associate with a compact set $C \subset \mathbb{R}^d$ is typically not larger than $t^{d/(d+2)} |B(0, R_0)|$ under $Q_t^{\mu, \nu}$.

The first result of this paper is that $|W_t^C|$ under $Q_t^{\mu, \nu}$ asymptotically equals to $t^{d/(d+2)} |B(0, R_0)|$ in the sense of the weak law of large numbers:

Theorem 2 *Let $d \geq 2, \mu \geq 0, \nu \geq 0$ and $\mu + \nu > 0$. Then we have for any nonpolar compact set C and $\epsilon > 0$,*

$$\lim_{t \rightarrow \infty} Q_t^{\mu, \nu}(|t^{-\frac{d}{d+2}} |W_t^C| - \omega_d R_0(d, \mu + \nu)^d| > \epsilon) = 0. \tag{5}$$

Moreover, if $\mu = 0, \nu > 0$ and $C \subset -K$, the law of $t^{-d/(d+2)} |W_t^C|$ under $Q_t^{0, \nu}$ satisfies following large deviation principle:

$$\begin{aligned} - \inf_{x \in \Gamma^o} I(x) &\leq \liminf_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0, \nu}(t^{-\frac{d}{d+2}} |W_t^C| \in \Gamma) \\ &\leq \limsup_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0, \nu}(t^{-\frac{d}{d+2}} |W_t^C| \in \Gamma) \leq - \inf_{x \in \bar{\Gamma}} I(x), \end{aligned} \tag{6}$$

where Γ is arbitrary Borel subset of $(0, \infty)$ and rate function I is given by

$$I(x) = \nu x + \lambda_d \left(\frac{\omega_d}{x} \right)^{\frac{2}{d}} - c(d, \nu).$$

Remark The assumption $C \subset -K$ may look rather technical. But the large deviation principle with above rate function fails when C is much larger than K . We shall give an example after the proof of Theorem 2.

Theorem 2, combined with Theorem 1, implies that the Wiener sausage under $Q_t^{\mu, \nu}$ covers almost all area of the ball in which it is confined.

The next result is the improvement of the convergence to L^p sense. We derive it as a corollary of following exponential tightness estimate:

Theorem 3 *Let $d \geq 2, \mu \geq 0, \nu \geq 0, \mu + \nu > 0$. Then for any $\eta > 0$,*

$$\sup_{t \geq 1} Q_t^{\mu, \nu} \left(\exp \left\{ \eta t^{-\frac{1}{d+2}} \sup_{0 \leq s \leq t} |Z_t| \right\} \right) < \infty.$$

Corollary 1 *Under the same conditions as in Theorem 3, we have*

$$\sup_{t \geq 1} Q_t^{\mu, \nu} \left(\exp \left\{ \eta \left(t^{-\frac{d}{d+2}} |W_t^C| \right)^{\frac{1}{d}} \right\} \right) < \infty \tag{7}$$

for any $\eta > 0$. Consequently, we have for all $p > 0$

$$\lim_{t \rightarrow \infty} Q_t^{\mu, \nu} \left(\left| t^{-\frac{d}{d+2}} |W_t^C| - \omega_d R_0(d, \mu + \nu)^d \right|^p \right) = 0. \tag{8}$$

Proof Since $W_t^C \subset B(0, \sup_{0 \leq s \leq t} |Z_s| + \text{diam}C)$, (7) follows directly from (3). From (7), we have that $\{t^{-dp/(d+2)} |W_t^C|^p\}_{t \geq 0}$ is uniformly integrable for any $p > 0$, which implies (8). \square

Now let us briefly explain the construction of this article. We start by considering exponential moments of $|W_t^C|$. Since negative exponential moments are easy to estimate, lower estimate of (5) follows from rather simple calculations. In the special case $\mu = 0, \nu > 0$ and $C \subset -K$, the upper bound for exponential moments can be extended to positive parameters. Then, we can derive large deviation upper bound using a similar argument to the Gärtner-Ellis theorem. The large deviation lower bound is obtained by considering a specific strategy for Wiener sausage to achieve given volume. Next, we shall give the proof of Theorem 3. Our strategy is essentially the same as the Povel’s proof of Theorem 1 but we need quantitatively refined estimate for the probability of the process exiting the confinement ball. The proofs of Theorem 1 and the upper estimate of (5) will also be given along the way in order to make this article reasonably self-contained.

2 Lower Estimate of Theorem 2 and Large Deviation

In this section, we are going to show the lower estimate of Theorem 2 and the large deviation result. Firstly, note that we can prove

$$S_t^{\mu, \nu} = \exp\{-c(d, \mu + \nu)t^{\frac{d}{d+2}}(1 + o(1))\} \tag{9}$$

when t goes to ∞ . Indeed, the lower bound is obvious since $S_t^{\mu, \nu}$ is bounded from below by $S_t^{0, \mu + \nu}$ with hard obstacle $(\text{supp } W) \cup K$. And the upper bound follows from the same argument as in the proof of Theorem 4.5.6 in [7], using the method of enlargement of obstacles which will be explained in Sect. 3.1. (In the upper bound of Theorem 4.5.6 in [7], the ‘enlarged obstacles’ are mainly considered and therefore the shape of obstacles has little to do with the argument.)

Similarly, for all $\lambda < 0$ and large enough t we can prove

$$\begin{aligned} & \mathbb{E}_\mu^1 \otimes \mathbb{E}_\nu^2 \otimes E_0 \left[\exp \left\{ - \int_0^t V(Z_s, \omega_1) ds + \lambda |W_t^C| \right\}; H_{S(\omega_2)} > t \right] \\ &= \mathbb{E}_\mu^1 \otimes \mathbb{E}_\nu^2 \otimes \mathbb{E}_{-\lambda}^3 \otimes E_0 \left[\exp \left\{ - \int_0^t V(Z_s, \omega_1) ds \right\}; H_{S(\omega_2) \cup \tilde{S}(\omega_3)} > t \right] \\ &= \exp\{-c(d, \mu + \nu - \lambda)t^{\frac{d}{d+2}}(1 + o(1))\}, \end{aligned} \tag{10}$$

where we set $\tilde{S}(\omega) = \bigcup_i (x_i - C)$ for $\omega = \sum_i \delta_{x_i}$. As a consequence,

$$\begin{aligned} & Q_t^{\mu, \nu}(\exp\{\lambda |W_t^C|\}) \\ &= \exp\{(c(d, \mu + \nu) - c(d, \mu + \nu - \lambda))t^{\frac{d}{d+2}}(1 + o(1))\} \end{aligned} \tag{11}$$

for $\lambda < 0$ when $t \rightarrow \infty$. Here we have implicitly used the fact $c(d, \mu + \nu) \neq c(d, \mu + \nu - \lambda)$ to ensure that the $o(1)$ in (9) and (10) is again $o(1)$ in (11). Now we prove the lower estimate of (5) in Theorem 2.

Proposition 1 For any $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} Q_t^{\mu, \nu} \left(t^{-\frac{d}{d+2}} |W_t^C| < \omega_d R_0(d, \mu + \nu)^d - \epsilon \right) = 0. \tag{12}$$

Proof Let $m = \omega_d R_0(d, \mu + \nu)^d - \epsilon$. Then Chebyshev’s inequality and (11) yield

$$\begin{aligned} & Q_t^{\mu, \nu} \left(t^{-\frac{d}{d+2}} |W_t^C| \leq m \right) \\ & \leq \exp \left\{ (-\lambda m + c(d, \mu + \nu) - c(d, \mu + \nu - \lambda)) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \\ & \leq \exp \left\{ \lambda \left(\frac{c(d, \mu + \nu) - c(d, \mu + \nu - \lambda)}{\lambda} - m \right) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \end{aligned} \tag{13}$$

as $t \rightarrow \infty$, provided $(c(d, \mu + \nu) - c(d, \mu + \nu - \lambda))/\lambda - m \neq 0$. If we note the fact that

$$\omega_d R_0(d, \nu)^d = \omega_d^{\frac{2}{d+2}} \nu^{-\frac{d}{d+2}} \left(\frac{2\lambda_d}{d} \right)^{\frac{d}{d+2}} = \frac{\partial}{\partial \nu} c(d, \nu), \tag{14}$$

we can actually find a $\lambda < 0$ such that $(c(d, \mu + \nu) - c(d, \mu + \nu - \lambda))/\lambda - m > 0$. Since the right hand side of (13) goes to 0 for this λ , we have done. \square

Next, we shall prove large deviation result. We start with the upper bound.

Proposition 2 Suppose $\mu = 0, \nu > 0$, and $C \subset -K$. Then for arbitrary Borel subset $\Gamma \subset (0, \infty)$,

$$\limsup_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0, \nu} \left(t^{-\frac{d}{d+2}} |W_t^C| \in \Gamma \right) \leq - \inf_{x \in \Gamma} I(x).$$

Here I is the rate function defined in Theorem 2.

Proof By the assumption $C \subset -K$, we can extend the upper bound of (11) to $0 < \lambda \leq \nu$ as follows:

$$\begin{aligned} Q_t^{0, \nu} \left(\exp \{ \lambda |W_t^C| \} \right) & \leq \frac{E_0[\exp \{ \lambda |W_t^{-K}| - \nu |W_t^{-K}| \}]}{E_0[\exp \{ -\nu |W_t^{-K}| \}]} \\ & = \exp \left\{ (c(d, \nu) - c(d, \nu - \lambda)) t^{\frac{d}{d+2}} (1 + o(1)) \right\}. \end{aligned}$$

Here we have used $c(d, \nu) - c(d, \nu - \lambda) \neq 0$ as in the derivation of (11). Therefore, we have following upper bound on the logarithmic generating function:

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0, \nu} \left(\exp \{ \lambda |W_t^C| \} \right) \\ & \leq \begin{cases} c(d, \nu) - c(d, \nu - \lambda) & (\lambda \leq \nu), \\ \infty & (\lambda > \nu). \end{cases} \end{aligned} \tag{15}$$

Then, the large deviation upper bound follows from very similar argument to the proof of the Gärtner-Ellis theorem (cf. [2]) and the rate function is given by the Fenchel-Legendre transform of the right hand side of (15). \square

Next, we go on to the lower bound, which do not require the assumption $C \subset -K$.

Proposition 3 *Suppose $\mu = 0$ and $\nu > 0$. Then for arbitrary Borel subset $\Gamma \subset (0, \infty)$,*

$$\liminf_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0,\nu} \left(t^{-\frac{d}{d+2}} |W_t^C| \in \Gamma \right) \geq - \inf_{x \in \Gamma^o} I(x).$$

Here I is the rate function defined in Theorem 2.

Proof It is enough to prove

$$\liminf_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0,\nu} \left(t^{-\frac{d}{d+2}} |W_t^C| \in (x - \delta, x + \delta) \right) \geq -I(x) \tag{16}$$

for any $x > 0$ and $\delta > 0$. To this end, set $r(t) = t^{1/(d+2)}(x/\omega_d)^{1/d}$ and consider the specific event

$$A_1 \times A_2 = \{\omega(B(0, r(t) + a)) = 0\} \times \{T_{B(0,r(t))} > t\}$$

where a was defined in (4). (Note that $C \subset -K \subset \bar{B}(0, a)$.) Since we know

$$\begin{aligned} Q_t^{0,\nu}(A_1 \times A_2) &= \frac{1}{S_{0,\nu}^{d+2}} \mathbb{P}_\nu(A_1) P_0(A_2) \\ &\geq \text{const}(d) \exp \left\{ - \left(\nu \omega_d r(t)^d + \frac{\lambda_d}{r(t)^2} t - c(d, \nu) t^{\frac{d}{d+2}} \right) (1 + o(1)) \right\} \\ &= \text{const}(d) \exp \left\{ -I(x) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \end{aligned}$$

from (3) and a well known eigenfunction expansion, (16) follows once we have shown

$$\lim_{t \rightarrow \infty} P_0 \left(t^{-\frac{d}{d+2}} |W_t^C| \in (x - \delta, x + \delta) \mid A_2 \right) = 1. \tag{17}$$

Proof of (17) Using Brownian scaling by the scale $\epsilon = t^{-1/(d+2)}$, (17) is equivalent to

$$\lim_{t \rightarrow \infty} P_0 \left(|W_{t\epsilon^2}^{eC}| \in (x - \delta, x + \delta) \mid T_{B(0,r(1))} > t\epsilon^2 \right) = 1. \tag{18}$$

To show this we shall use Theorem 3.2.3 in [7], which claims

$$\lambda(U \setminus K) \geq \lambda(U) + \frac{\mu(U) - \lambda(U)}{\mu(U)} \inf_K \varphi_U^2 \cdot \text{cap}_U(K)$$

for any non-empty bounded domain $U \subset \mathbb{R}^d$ and compact set $K \subset U$. Here φ_U denotes the positive L^2 -normalized principal eigenfunction and $\mu(U)$ the second smallest eigenvalue both associated with $-\Delta/2$ in U with Dirichlet boundary condition. From this and the fact that $\varphi_{B(0,r)}$ has nondegenerate gradient near the boundary, we have for all $x \in B(0, r(1) - \sqrt{\epsilon})$,

$$\begin{aligned} &\log P_0(H_{x-\epsilon C} > t\epsilon^2 \mid T_{B(0,r(1))} > t\epsilon^2) \\ &= -(\lambda(B(0, r(1)) \setminus (x - \epsilon C)) - \lambda(B(0, r(1)))) t^{\frac{d}{d+2}} (1 + o(1)) \\ &\leq -\text{const}(d, r(1)) \epsilon \cdot \text{cap}_{B(0,r(1))}(\epsilon C) t^{\frac{d}{d+2}} (1 + o(1)) \end{aligned} \tag{19}$$

as $t \rightarrow \infty$. The right hand side of (19) goes to $-\infty$ as $t \rightarrow \infty$ since

$$\text{cap}_{B(0,r(1))}(\epsilon C) \sim \begin{cases} \text{const}(C, 2)(\log \frac{1}{\epsilon})^{-1} & (d = 2), \\ \text{const}(C, d)\epsilon^{d-2} & (d \geq 3). \end{cases}$$

As a consequence, we have

$$P_0(H_{x-\epsilon C} > t\epsilon^2 | T_{B(0,r(1))} > t\epsilon^2) \rightarrow 1 \quad (t \rightarrow \infty)$$

and therefore

$$\begin{aligned} & E_0[|W_{t\epsilon^2}^{\epsilon C}| | T_{B(0,r(1))} > t\epsilon^2] \\ &= \int_{\mathbb{R}^d} P_0(H_{x-\epsilon C} \leq t\epsilon^2 | T_{B(0,r(1))} > t\epsilon^2) dx \\ &\geq \int_{B(0,r(1)-\sqrt{\epsilon})} P_0(H_{x-\epsilon C} \leq t\epsilon^2 | T_{B(0,r(1))} > t\epsilon^2) dx \\ &\rightarrow |B(0, r(1))| = x \quad (t \rightarrow \infty). \end{aligned}$$

This, together with the obvious fact that

$$P_0(|W_{t\epsilon^2}^{\epsilon C}| \leq |B(0, r(1) + a\epsilon)| | T_{B(0,r(1))} > t\epsilon^2) = 1,$$

implies (18). □

Now that we have shown (17), the proof of Proposition 3 is completed. □

Finally, we shall give an example noticed after Theorem 2. Basically, it comes from the case where the ‘ballistic strategy’ dominates above ‘localizing strategy’.

Example Let $C = \overline{B}(0, R)$ and $K = \overline{B}(0, 1)$ where R will be taken large. The key is to consider the specific strategy:

$$A = \left\{ Z_t \in B\left(t^{\frac{d}{d+2}}h, rt^{\frac{d}{d+2}}\right) \right\}$$

where $h \in \mathbb{R}^d$ and $r > 0$ satisfy $|h| > r$. On this event, we have

$$|W_t^C| > t^{d/(d+2)}(|h| - r)\omega_{d-1}R^{d-1} \tag{20}$$

by considering cross sections orthogonal to h . For the ‘cost’ of this strategy, we use the large deviation estimate

$$t^{-\frac{d}{d+2}} \log Q_t^{0,v}(A) \sim - \inf_{x \in B(h,r)} \beta_0(x) \quad (t \rightarrow \infty) \tag{21}$$

which was shown by Sznitman in [6]. Here β_0 is the annealed Lyapunov exponent introduced in [6], which measures the decay rate of the probability for Brownian motion to perform a long crossing among $S(\omega)$.

Combining (20) and (21), we have

$$\begin{aligned} &\liminf_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{0,v} (t^{-d/(d+2)} |W_t^C| > (|h| - r)\omega_{d-1} R^{d-1}) \\ &\geq - \inf_{x \in B(h,r)} \beta_0(x). \end{aligned}$$

Since the right hand side is independent of R , the upper bound of (6) breaks down when R is large.

3 Upper Estimates

We shall prove the upper estimate of (5) and Theorem 3 in this section. As described at the end of the Sect. 1, the proofs of them are based on Theorem 1 and its proof. The starting point is to adopt the scale

$$\epsilon = t^{-\frac{1}{d+2}}$$

and consider $\epsilon^{-1}Z_{t\epsilon^2}$, $\mathbb{P}_{\mu\epsilon^{-d}}^1$ and $\mathbb{P}_{v\epsilon^{-d}}^2$. We introduce the notation $\mathbf{E}_\epsilon = \mathbb{E}_{\mu\epsilon^{-d}}^1 \otimes \mathbb{E}_{v\epsilon^{-d}}^2$ for simplicity. Then, for instance, Theorem 1 follows once we have shown that for all $(\omega_1, \omega_2) \in \Omega^2$ there exists a ball $B(\omega_1, \omega_2)$ with radius in $[R_0, R_0 + \kappa_1\epsilon^{\kappa_2}]$ and center in $B(0, R_0 + \kappa_1\epsilon^{\kappa_2})$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{S_t^{\mu,v}} \mathbf{E}_\epsilon \otimes E_0 \left[\exp \left\{ - \int_0^\tau V_\epsilon(Z_s, \omega_1) ds \right\}; T_{B(\omega_1, \omega_2)} \wedge H_{S_\epsilon(\omega_2)} > \tau \right] = 1.$$

Here $\tau = t\epsilon^2$ and

$$\begin{aligned} V_\epsilon(x, \omega) &= \epsilon^{-2} \sum_i W \left(\frac{x - x_i}{\epsilon} \right), \\ S_\epsilon(\omega) &= \bigcup_i (x_i + \epsilon K) \end{aligned}$$

for $\omega = \sum_i \delta_{x_i}$. For $(\omega_1, \omega_2) \in \Omega^2$ and open set U we also define

$$\lambda_{\omega_1, \omega_2}^\epsilon(U) = \lambda_{V_\epsilon(\cdot, \omega_1)}(U \setminus S_\epsilon(\omega_2))$$

where $\lambda_V(U)$ denotes the principal Dirichlet eigenvalue of $-1/2\Delta + V$ in U .

3.1 Method of Enlargement of Obstacles

In this subsection, we shall recall elements and some estimates from the method of enlargement of obstacles in [7]. The method is based on coarse graining of the space and construction of two disjoint sets $\mathcal{D}_\epsilon(\omega)$ and $\mathcal{B}_\epsilon(\omega)$ for $\omega \in \Omega$ and $\epsilon \in (0, 1)$. The set $\mathcal{D}_\epsilon(\omega)$ is called ‘density set’, where one enlarge (both the support and the height of) the obstacles and the set $\mathcal{B}_\epsilon(\omega)$ is called ‘bad set’, where the obstacles exist but are left almost untouched.

To construct these sets, we need parameters $0 < \alpha < \gamma < \beta < 1$, $\delta > 0$ and an integer $L \geq 2$. Using these parameters, we introduce spatial scales $1 \gg \epsilon^\alpha \gg \epsilon^\gamma \gg \epsilon^\beta \gg \epsilon$ and

L -adic decomposition of \mathbb{R}^d . We also need following notation concerning L -adic decomposition of \mathbb{R}^d . Let \mathcal{I}_k be the collection of indices of the form

$$\mathbf{i} = (i_0, i_1, \dots, i_k) \in \mathbb{Z}^d \times (\{0, 1, \dots, L - 1\}^d)^k.$$

We associate to above index \mathbf{i} a box:

$$C_{\mathbf{i}} = q_{\mathbf{i}} + L^{-k}[0, 1]^d \quad \text{where } q_{\mathbf{i}} = i_0 + L^{-1}i_1 + \dots + L^{-k}i_k.$$

For $\mathbf{i} \in \mathcal{I}_k$ and $k' \leq k$, we define the truncation

$$[\mathbf{i}]_{k'} = (i_0, i_1, \dots, i_{k'}).$$

Finally, we pick integers $n_{\alpha}(\epsilon)$, $n_{\gamma}(\epsilon)$ and $n_{\beta}(\epsilon)$ such that

$$L^{-n_*-1} \leq \epsilon^* < L^{-n_*} \quad (* \text{ is } \alpha, \gamma \text{ or } \beta). \tag{22}$$

L^{-n_*} plays the role of the scale ϵ^* in the context of L -adic decomposition.

For $\omega = \sum_q \delta_{x_q}$ and $\mathbf{i} \in \mathcal{I}_k$,

$$K_{\mathbf{i}} = L^k \left(\bigcup_{x_q \in C_{\mathbf{i}}} \overline{B}(x_q, a\epsilon) \right) \tag{23}$$

is called the skeleton of traps. Here a is taken so large as $\overline{B}(0, a)$ includes both K and $\text{supp } W$. (We take larger a so that $\overline{B}(0, a)$ includes $-C$ for the proof of (10).) Using this skeleton, the density set $\mathcal{D}_{\epsilon}(\omega)$ is defined as follows:

Definition 1 ((4.2.13) in [7]) $C_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}_{n_{\gamma}}$) is called a density box if it satisfies the quantitative Wiener criterion:

$$\sum_{n_{\alpha} < k \leq n_{\gamma}} \text{cap}(K_{[\mathbf{i}]_k}) \geq \delta(n_{\gamma} - n_{\alpha}).$$

Here $\text{cap}(\cdot)$ denotes the capacity relative to $1 - \Delta/2$ when $d = 2$ and $-\Delta/2$ when $d \geq 3$. The union of all density boxes is denoted by $\mathcal{D}_{\epsilon}(\omega)$.

Next, the bad set $\mathcal{B}_{\epsilon}(\omega)$ is defined as follows:

Definition 2 ((4.3.47) in [7]) $C_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}_{n_{\beta}}$) is called a bad box if $\omega(C_{\mathbf{i}}) \geq 1$ and $C_{\mathbf{i}} \not\subset \mathcal{D}_{\epsilon}(\omega)$. The union of all bad boxes is denoted by $\mathcal{B}_{\epsilon}(\omega)$.

As a result of above construction, $\mathcal{D}_{\epsilon}(\omega)$ and $\mathcal{B}_{\epsilon}(\omega)$ satisfy

$$\begin{aligned} \mathcal{D}_{\epsilon}(\omega) \cap \mathcal{B}_{\epsilon}(\omega) &= \emptyset, \\ \omega(\mathbb{R}^d \setminus (\mathcal{D}_{\epsilon}(\omega) \cup \mathcal{B}_{\epsilon}(\omega))) &= 0, \end{aligned}$$

for each box $C_q = q + [0, 1]^d$, $q \in \mathbb{Z}^d$, the sets $\mathcal{D}_{\epsilon}(\omega) \cap C_q$ (resp. $\mathcal{B}_{\epsilon}(\omega) \cap C_q$) can take no more than $2^{\epsilon^{-d\gamma}}$ (resp. $2^{\epsilon^{-d\beta}}$) different shapes as ω varies over Ω .

The notation and definitions are exactly the same as in [7] so far.

Next, we state four estimates which is the mixed obstacles version of the results in [7]. For the proofs of these estimates, we shall put some comments at the end of this subsection.

We define the density set $\mathcal{D}_\epsilon(\omega_1, \omega_2)$ and the bad set $\mathcal{B}_\epsilon(\omega_1, \omega_2)$ as above by letting $\omega = \omega_1 + \omega_2$. The first claims that solidifying $\overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2)$, i.e. imposing Dirichlet conditions on $\overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2)$, does not cause essential increase of the principal eigenvalues.

Spectral control I (Theorem 4.2.3 in [7]) There exist $c_1(d, W, K) > 0$ such that for all $\rho \in (0, \delta c_1 \frac{\gamma - \alpha}{(d+2) \log L})$ and $M > 0$,

$$\lim_{\epsilon \rightarrow 0} \sup_{(\omega_1, \omega_2) \in \Omega^2, U: \text{open}} \epsilon^{-\rho} (\lambda_{\omega_1, \omega_2}^\epsilon(U \setminus \overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2)) \wedge M - \lambda_{\omega_1, \omega_2}^\epsilon(U) \wedge M) = 0.$$

The second corresponds to the volume of the bad set. Since we cannot control the solidifying effect on $\mathcal{B}_\epsilon(\omega_1, \omega_2)$, we need to show that it is not too large.

Volume control (Theorem 4.3.6 in [7]) There exist $L \geq 2, \delta > 0$ and $\kappa > 0$ such that

$$\lim_{\epsilon \rightarrow 0} \sup_{\omega \in \Omega, q \in \mathbb{Z}^d} \epsilon^{-\kappa} |\mathcal{B}_\epsilon(\omega_1, \omega_2) \cap C_q| < 1.$$

The third estimate says that the region where $\mathcal{D}_\epsilon(\omega_1, \omega_2)^c$ is locally thin is hard to survive. Therefore one can expect that, as in the first estimate, solidifying such a region does not cause essential increase of the principal eigenvalues. This is precisely the role of the fourth estimate.

Spectral control II (Proposition 4.2.4 in [7]) There exist $c_2(d) > 0$ such that for all $\epsilon \in (0, 1), r \in (0, 1/4), (\omega_1, \omega_2) \in \Omega^2$ and open set U satisfying:

$$\begin{aligned} 4a\epsilon &< L^{-n_\gamma(\epsilon)} < L^{-n_\alpha(\epsilon)} < r, \\ \delta c_1(n_\gamma(\epsilon) - n_\alpha(\epsilon)) &> \log 2, \\ \sup_{q \in \mathbb{Z}^d} |(U \setminus \overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2)) \cap C_q| &< r^d, \end{aligned}$$

one has

$$\lambda_{\omega_1, \omega_2}^\epsilon(U) > \frac{c_2}{r^2}.$$

Spectral control III (Theorem 4.2.6 in [7]) For all $M > 0$, there exist $c_3(d) > 0, c_4(d, M) > 1$ and $r_0(d, M) \in (0, 1/4)$ such that

$$\limsup_{\epsilon \rightarrow 0} \widetilde{\sup} \exp \left\{ c_3 \left[\frac{R}{4r} \right] \right\} (\lambda_{\omega_1, \omega_2}^\epsilon(U_1) \wedge M - \lambda_{\omega_1, \omega_2}^\epsilon(U_2) \wedge M) \leq 1, \tag{24}$$

where $[\cdot]$ denotes the integer part and $\widetilde{\sup}$ the supremum over all $(\omega_1, \omega_2) \in \Omega^2, U_1 \subset U_2$: open, $R > 0$ and $r > 0$ such that for some closed set A ,

$$\begin{aligned} L^{-n_\alpha(\epsilon)} &< r < r_0, \\ \frac{R}{4r} &> c_4, \end{aligned} \tag{25}$$

$$\begin{aligned} \sup_{q \in \mathbb{Z}^d} |(U_2 \setminus (A \cup \overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2))) \cap C_q| &< r^d, \\ \text{dist}_{\|\cdot\|}(U_2 \setminus U_1, A \cap U_2) &\geq R. \end{aligned} \tag{26}$$

Here, $\|\cdot\|$ is the maximal norm on \mathbb{R}^d .

For a typical case (24) applies, we introduce

$$\mathcal{A}_\epsilon(\omega_1, \omega_2) = \bigcup_{C_q: |C_q \setminus \mathcal{D}_\epsilon(\omega_1, \omega_2)| \geq r^d} \bar{C}_q,$$

$$\mathcal{O}_\epsilon(\omega_1, \omega_2) = \{x \in \mathbb{R}^d; \text{dist}_{\|\cdot\|}(x, \mathcal{A}_\epsilon(\omega_1, \omega_2)) < R\}.$$

Each C_q contained in $\mathcal{A}_\epsilon(\omega_1, \omega_2)$ is called *clearing box* and $\mathcal{A}_\epsilon(\omega_1, \omega_2)$ is called *clearing set*. Then for any open set \mathcal{T} and $R > 0$,

$$U_1 = \mathcal{T} \cap \mathcal{O}_\epsilon(\omega_1, \omega_2), \quad U_2 = \mathcal{T}$$

satisfy (25) and (26) with $A = \mathcal{A}_\epsilon(\omega_1, \omega_2)$.

Before closing this subsection, we briefly explain how to prove these estimates. The volume estimate is equivalent to that in [7] since our bad set is the same as that for hard obstacles with $K = \bar{B}(0, a)$. For the spectral controls, we mention that in the proofs in [7], the dependence on the shapes of obstacles only appears in following key lemma:

Key Lemma (Lemma 4.2.1 in [7]) There exist $c_1(d, W) > 0$ (or $c_1(d, K)$ for hard obstacles) such that if $4a\epsilon < L^{-n_\gamma}$,

$$E_x \left[\exp \left\{ - \int_0^{H_{n_\alpha}(\epsilon)} V_\epsilon(Z_s, \omega) ds \right\} \right] \leq \exp \left\{ -c_1 \sum_{n_\alpha < k \leq n_\gamma} \text{cap}(K_{[i]_k}) \right\}$$

$$\left(\text{resp. } P_x(H_{n_\alpha} < H_{S(\omega)}) \leq \exp \left\{ -c_1 \sum_{n_\alpha < k \leq n_\gamma} \text{cap}(K_{[i]_k}) \right\} \right)$$

for any $\omega \in \Omega$, $i \in \mathcal{I}_{n_\gamma}$, and $x \in C_i$. Here $H_{n_\alpha} = \inf\{s \geq 0; \|Z_s - Z_0\| \geq L^{-n_\alpha}\}$.

However, it is routine to extend this key lemma to our mixed obstacles if we replace c_1 by $c_1(d, W) \wedge c_1(d, K)$. (This is the constant $c_1(d, W, K)$ appeared in spectral control I and II.) Thus we can prove the spectral controls in our setting by exactly the same ways as in [7].

3.2 Construction of the Confinement Ball

In this subsection, we shall construct $B(\omega_1, \omega_2)$ in Theorem 1. Since results are essentially the same as in [4], we omit the proofs and refer counterparts instead. From now on, we fix an admissible collection of parameters

$$\alpha, \beta, \gamma, \delta, L, \rho, \kappa$$

and pick $R = 1$ and

$$r = \epsilon^{\alpha_0}, \quad \text{with } 0 < \alpha_0 < \min \left(\alpha, 1 - \beta, \frac{\kappa}{d} \right) < 1$$

which allow us to apply the results in Sect. 3.1. For an explanation about the admissible collection of parameters, we refer reader to the remarks after (4.3.66) in [7].

Now let us start by introducing the open set

$$\mathcal{T} = (-[t], [t])^d$$

and notation

$$v(\tau) = \exp\left\{-\int_0^\tau V_\epsilon(Z_s, \omega_1) ds\right\} 1_{\{H_{S_\epsilon}(\omega_2) > \tau\}}.$$

Then, using standard estimates on Brownian motion and (9), we have

$$\frac{1}{t} \log \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); T_T \leq \tau] < 0 \tag{27}$$

for large t . Since this is good enough for our purpose, we restrict our consideration on $\{T_T > \tau\}$ in the sequel. We also introduce the open set

$$\mathcal{U}(\omega_1, \omega_2) = (T \cap \mathcal{O}_\epsilon(\omega_1, \omega_2)) \setminus \overline{\mathcal{D}}_\epsilon(\omega_1, \omega_2).$$

Then, we have following constraint on this set.

Proposition 4 (Proposition 1 in [4]) *Pick $\chi \in (0, 1)$ such that*

$$\chi > \max\left(\beta + \alpha_0, 1 - \left(\frac{\kappa}{d} - \alpha_0\right), 1 - \frac{\rho}{d}\right)$$

and let

$$\begin{aligned} \alpha_1 &< \min(d(1 - \chi), 1), \\ c_5(a, d, \mu + \nu) &> 2(1 + \gamma(a, d, \mu + \nu)), \end{aligned}$$

where γ and a were introduced in (3) and (4), respectively. Then we have

$$\begin{aligned} &\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); T_T > \tau, (\mu + \nu)|\mathcal{U}| + \lambda(\mathcal{U}) > c(d, \mu + \nu) + c_5\epsilon^{\alpha_1}] \\ &\leq \exp\left\{-(1 + \gamma)t^{\frac{d - \alpha_1}{d + 2}}\right\} \end{aligned}$$

as $t \rightarrow \infty$.

If we set

$$E = \{(\mu + \nu)|\mathcal{U}| + \lambda(\mathcal{U}) \leq c(d, \mu + \nu) + c_5\epsilon^{\alpha_1}, \lambda_{\omega_1, \omega_2}^\epsilon(T) \leq 2c(d, \mu, \nu)\}, \tag{28}$$

then we have

$$\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); T_T > \tau, E^c] \leq \exp\left\{-(1 + \gamma)t^{\frac{d - \alpha_1}{d + 2}}\right\}$$

from Proposition 4.

The next proposition says that \mathcal{U} is, in a ‘measurable sense’, close to an optimal ball of the variational problem in (1).

Proposition 5 (Proposition 2 in [4]) *For any $(\omega_1, \omega_2) \in E$, there exists a ball B with radius*

$$R = \left(\frac{|\mathcal{U}|}{\omega_d}\right)^{\frac{1}{d}}$$

such that for large t ,

$$|\mathcal{U} \setminus B| \leq c_6 \epsilon^{\frac{\alpha_1}{20}}, \quad R \leq R_0(d, \mu + \nu) + c_6 \epsilon^{\frac{\alpha_1}{2}} \tag{29}$$

where R_0 was defined in (2) and $c_6(d, \mu + \nu) > 0$ is a constant.

Thanks to Proposition 5, we can introduce for $(\omega_1, \omega_2) \in E$,

$$B_l : \text{the concentric ball to } B \text{ with radius } R_l = R_0 + l \tag{30}$$

where l may depends on t and is assumed to satisfy

$$l \geq \epsilon^{\alpha_2}, \quad \alpha_2 \in \left(0, \alpha_0 \wedge \frac{\alpha_1}{20d}\right).$$

It should be pointed out that the ball B , and thus B_l , corresponding to a configuration $(\omega_1, \omega_2) \in E$ need not be unique. Since only $B_l \ni 0$ matters in the sequel, we define $B^{(l)}(\omega_1, \omega_2)$ as a B_l which contains the origin when (ω_1, ω_2) belongs to

$$\Omega_1 = \{\text{There exists a ball } B_l \text{ with } 0 \in B_l\} \cap E,$$

and otherwise $B^{(l)}(\omega_1, \omega_2) = B(0, R_0 + l)$. In particular, we take $l = \epsilon^{\alpha_2}$ for Theorem 1. Then it follows from (29) that the radius R_l of B_l satisfies

$$\begin{aligned} R_0 \leq R_l &\leq R_0 + c_6 \epsilon^{\frac{\alpha_1}{2}} + \epsilon^{\alpha_2} \\ &\leq R_0 + (c_6 + 1)\epsilon^{\alpha_2} \end{aligned}$$

and therefore $B^{(l)}(\omega_1, \omega_2)$ has the properties stated in Theorem 1 with

$$\kappa_1 = c_6 + 1, \quad \kappa_2 = \alpha_2.$$

3.3 Control of the Excursion Probability

In this subsection, we shall derive upper bounds on the probability of the excursion of the surviving process from B to B_l^c :

Proposition 6 *There exists a constant $c_7(a, d, \mu + \nu) > 0$ such that for E introduced in (28), any B_l defined in (30) corresponding to $(\omega_1, \omega_2) \in \Omega^2$ with $l \leq \tau$ and large t , we have*

$$\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E, T_T > \tau, T_{B_l} \leq \tau] \leq \exp\left\{-c_7 l t^{\frac{\alpha_3}{d+2}}\right\}$$

where $\alpha_3 = \alpha_0 \wedge (\alpha_1/20d)$.

This proposition is the (slightly refined) quantitative version of Proposition 3 in [4]. The proof will involve estimates on the probability of two types of events. These events display two possibilities for surviving Brownian motion after it exits B_l : either it returns to \bar{B} immediately or stays outside \bar{B} certain amount of time. To deal with these events, we shall use two lemmas. The first lemma implies that the complement of B almost looks like the ‘forest set’.

Lemma 1 *Let $(\omega_1, \omega_2) \in E$ and define $c_8 = (c_6 + 1)^{1/d}$, $r_1(\epsilon) = c_8\epsilon^{\alpha_3}$. Then there exist constants $c_9(d) > 0$, $c_{10}(d) > 0$ such that for large t ,*

$$\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T} \setminus \overline{B}) \geq \frac{c_9}{l_1^2}, \tag{31}$$

$$\sup_{z \in B_f^c} E_z[v(H_{\overline{B}}); H_{\overline{B}} < T_{\mathcal{T}}] \leq \exp\left\{-c_{10} \frac{l}{r_1}\right\}. \tag{32}$$

Proof This lemma is essentially the same as Lemma 1 in [4] except for dealing with longer “return” in (32). The proof also goes in the same way and we omit the detail. \square

Equations (31) and (32) are related to the estimates on the probability of above events. But they are not enough, because we only have rough asymptotics for normalizing constant $S_t^{\mu, \nu}$ (cf. (9)). Therefore, we need some cancellation and the next lemma meets our need. (This is also essentially the same as Lemma 2 in [4] but we give the proof since it seems shorter and self-contained.)

Lemma 2

$$\sup_{t \geq 1} \frac{1}{|\mathcal{T}|^2 S_t^{\mu, \nu}} \mathbf{E}_\epsilon \left[\exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})\tau\} \right] < \infty.$$

Proof We start by introducing the notation

$$R_t^{U, V} f(x) = E_x \left[f(Z_t) \exp\left\{-\int_0^t V(Z_s) ds\right\}; T_U > t \right]$$

for a nonempty open set U and a nonnegative function V . If $|U| < \infty$ and V is locally bounded, then $R_t^{U, V}$ defines a self-adjoint trace class semigroup on $L^2(U, dx)$, see for instance (1.3.15) in [7]. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(U, dx)$. Using translation invariance with respect to \mathbb{P} , we find

$$\begin{aligned} S_t^{\mu, \nu} &= \mathbf{E}_\epsilon \otimes E_0 \left[\exp\left\{-\int_0^\tau V_\epsilon(Z_s, \omega_1) ds\right\}; H_{S_\epsilon(\omega_2)} > \tau \right] \\ &= \frac{1}{|\mathcal{T}|} \mathbf{E}_\epsilon \left[\int_{\mathcal{T}} E_x \left[\exp\left\{-\int_0^\tau V_\epsilon(Z_s, \omega_1) ds\right\}; H_{S_\epsilon(\omega_2)} > \tau \right] dx \right] \\ &\geq \frac{1}{|\mathcal{T}|} \mathbf{E}_\epsilon \left[\int_{\mathcal{T} \setminus S_\epsilon} E_x[v(\tau); T_{\mathcal{T}} > \tau] dx; \lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu) \right] \\ &= \frac{1}{|\mathcal{T}|} \mathbf{E}_\epsilon \left[\langle R_\tau^{\mathcal{T} \setminus S_\epsilon(\omega_2), V_\epsilon(\cdot, \omega_1)} 1, 1 \rangle; \lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu) \right] \\ &\geq \frac{1}{|\mathcal{T}|} \mathbf{E}_\epsilon \left[\langle \phi, 1 \rangle^2 \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})\tau\}; \lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu) \right] \end{aligned} \tag{33}$$

where ϕ is a normalized nonnegative eigenfunction associated with $\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})$. Here we have implicitly used that \mathcal{T} has finite volume and V_ϵ is locally bounded for all $(\omega_1, \omega_2) \in \Omega^2$. Since $\|R_s^{\mathcal{T} \setminus S_\epsilon, V_\epsilon}\|_{1 \rightarrow \infty} \leq (2\pi s)^{-\frac{d}{2}}$ for all $s > 0$, we have

$$0 \leq \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})\} \phi(z)$$

$$\begin{aligned}
 &= R_1^{\mathcal{T} \setminus S_\epsilon(\omega_2), V_\epsilon(\cdot, \omega_1)} \phi(z) \\
 &\leq (2\pi)^{-\frac{d}{2}} \langle \phi, 1 \rangle
 \end{aligned}$$

and therefore for $(\omega_1, \omega_2) \in \{\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu)\}$,

$$\begin{aligned}
 \langle \phi, 1 \rangle^2 &\geq (2\pi)^d \exp\{-4c(d, \mu + \nu)\} \sup_{z \in \mathbb{R}^d} \phi(z)^2 \\
 &\geq (2\pi)^d \exp\{-4c(d, \mu + \nu)\} |\mathcal{T}|^{-1}
 \end{aligned}$$

where we have used $\text{supp } \phi \subset \mathcal{T}$ together with $\|\phi\|_2 = 1$ in the second inequality. Coming back to (33), we have shown that for $t \geq 1$,

$$S_t^{\mu, \nu} \geq \frac{c_{11}(d, \mu + \nu)}{|\mathcal{T}|^2} \mathbf{E}_\epsilon \left[\exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})\tau\}; \lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu) \right].$$

We can drop $\{\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) \leq 2c(d, \mu + \nu)\}$ since

$$\begin{aligned}
 &\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \left[\exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T})\tau\}; \lambda_{\omega_1, \omega_2}^\epsilon(\mathcal{T}) > 2c(d, \mu + \nu) \right] \\
 &\leq \frac{1}{S_t^{\mu, \nu}} \exp\{-2c(d, \mu + \nu)\tau\} \rightarrow 0
 \end{aligned}$$

as $t \rightarrow \infty$ from (9). Thus the claim of Lemma 2 follows. □

We are now ready to prove Proposition 6.

Proof of Proposition 6 Let θ denote the canonical shift and define

$$\begin{aligned}
 \{T_{B_l} \leq \tau < T_{\mathcal{T}}\} &\subset \{2T_{B_l} \leq \tau < T_{\mathcal{T}}, H_{\overline{B}} \circ \theta_{T_{B_l}} > lr_1\} \\
 &\cup \{2T_{B_l} \leq \tau < T_{\mathcal{T}}, H_{\overline{B}} \circ \theta_{T_{B_l}} \leq lr_1\} \\
 &\cup \left\{ \frac{\tau}{2} < T_{B_l} \leq \tau, T_{\mathcal{T}} > \tau \right\} \\
 &= A_1 \cup A_2 \cup A_3
 \end{aligned}$$

for $t \geq 1$ and $(\omega_1, \omega_2) \in E$. Here note that our choice $l \leq \tau$ implies $\tau/2 + lr_1 < \tau$ for large t . We start with A_1 . Pick $(\omega_1, \omega_2) \in E$ and write

$$\begin{aligned}
 E_0[v(\tau); A_1] &= \sum_{k=0}^{\lfloor \tau/2 \rfloor} E_0[v(\tau); k \leq T_{B_l} < k + 1, A_1] \\
 &= \sum_{k=0}^{\lfloor \tau/2 \rfloor} E_0[v(\tau); E_k, A_1].
 \end{aligned} \tag{34}$$

Applying strong Markov property at $T_{B_l} + lr_1$ and T_{B_l} and using (3.1.9) of [7], we find for $1 \leq k \leq \lfloor \tau/2 \rfloor$ and large t that

$$\begin{aligned}
 & E_0[v(\tau); E_k, A_1] \\
 & \leq c(d)\left(1 + (\lambda_{\omega_1, \omega_2}^\epsilon(T)\tau)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - (k + 1 + lr_1))\} \\
 & \quad \times E_0[v(T_{B_l} + lr_1); E_k, H_{\bar{B}} \circ \theta_{T_{B_l}} > lr_1, T_T > T_{B_l}, T_T \circ \theta_{T_{B_l}} > lr_1] \\
 & \leq c(d)\left(1 + (2c(d, \mu + \nu)\tau)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - (k + 1 + lr_1))\} \\
 & \quad \times \sup_{z \in B^c} E_z[v(lr_1); H_{\bar{B}} > lr_1, T_T > lr_1] E_0[v(T_{B_l}); E_k, T_T > T_{B_l}] \\
 & \leq c(d)^2\left(1 + (2c(d, \mu + \nu)\tau)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - (k + 1 + lr_1))\} \\
 & \quad \times \left(1 + (\lambda_{\omega_1, \omega_2}^\epsilon(T \setminus \bar{B})lr_1)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T \setminus \bar{B})lr_1\} E_0[v(k); T_T > k] \\
 & \leq c_{12} \exp\left\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - lr_1) - \frac{1}{2}\lambda_{\omega_1, \omega_2}^\epsilon(T \setminus \bar{B})lr_1\right\} \tag{35}
 \end{aligned}$$

for some constant $c_{12}(d, \mu + \nu) > 0$. Here we have used $\lambda_{\omega_1, \omega_2}^\epsilon(T) \leq 2c(d, \mu + \nu)$ for $(\omega_1, \omega_2) \in E$ and

$$\lambda_{\omega_1, \omega_2}^\epsilon(T \setminus \bar{B})lr_1 \geq \frac{c_9 l}{r_1} \geq \frac{c_9}{c_8} t^{\frac{\alpha_3 - \alpha_2}{d+2}} \tag{36}$$

from (31) and our choice $l \geq t^{-\alpha_2/(d+2)}$. Coming back to (34) and using (36) again, we have

$$\begin{aligned}
 & \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E \cap A_1] \\
 & \leq c_{12} \tau^{d+1} |T|^2 \exp\left\{-\frac{c_9 l}{2r_1} + (c(d, \mu + \nu) + 1)lr_1\right\} \\
 & \quad \times \sup_{t \geq 1} \frac{1}{|T|^2 S_t^{\mu, \nu}} \mathbf{E}_\epsilon[\exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)\tau\}; E].
 \end{aligned}$$

Therefore for large t , Lemma 2 gives us

$$\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E \cap A_1] \leq \exp\left\{-c_9 \frac{l}{r_1}\right\}$$

with slightly smaller c_9 .

Next, we shall deal with A_2 . As in (34) we write

$$E_0[v(\tau); A_2] = \sum_{k=0}^{[\tau/2]} E_0[v(\tau); E_k, A_2], \tag{37}$$

for $(\omega_1, \omega_2) \in E$. Then we have for $0 \leq k \leq [\tau/2]$ and large enough t ,

$$\begin{aligned}
 & E_0[v(\tau); E_k, A_2] \\
 & \leq c(d)\left(1 + (\lambda_{\omega_1, \omega_2}^\epsilon(T)\tau)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - (k + 1 + lr_1))\} \\
 & \quad \times E_0[v(T_{B_l} + lr_1); E_k, H_{\bar{B}} \circ \theta_{T_{B_l}} \leq lr_1, T_T > T_{B_l}, T_T \circ \theta_{T_{B_l}} > lr_1] \\
 & \leq c(d)\left(1 + (2c(d, \mu + \nu)\tau)^{\frac{d}{2}}\right) \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - (k + 1 + lr_1))\}
 \end{aligned}$$

$$\begin{aligned} & \times \sup_{z \in B^c} E_z[v(H_{\bar{B}}); H_{\bar{B}} \leq lr_1 < T_{\mathcal{T}}] E_0[v(T_{B_l}); E_k, T_{\mathcal{T}} > T_{B_l}] \\ & \leq c_{13} \tau^d \sup_{z \in B^c} E_z[v(H_{\bar{B}}); H_{\bar{B}} < T_{\mathcal{T}}] \exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)(\tau - lr_1)\} \end{aligned} \tag{38}$$

for some constant $c_{13}(d, \mu + \nu) > 0$. Coming back to (37) and using (32), we find

$$\begin{aligned} & \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E \cap A_2] \\ & \leq c_{13} \tau^{d+1} |T|^2 \exp\left\{-\frac{c_{10}l}{r_1} + (c(d, \mu + \nu) + 1)lr_1\right\} \\ & \quad \times \sup_{t \geq 1} \frac{1}{|T|^2 S_t^{\mu, \nu}} \mathbf{E}_\epsilon[\exp\{-\lambda_{\omega_1, \omega_2}^\epsilon(T)\tau\}; E]. \end{aligned}$$

Therefore it follows as before that

$$\frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E \cap A_2] \leq \exp\left\{-c_{10} \frac{l}{r_1}\right\}$$

with slightly smaller c_{10} .

As for A_3 , observe that on $\{\tau/2 < T_{B_l} \leq \tau\}$ the reversed path starting at Z_τ exits B_l before time $\tau/2$. Since the estimates (35) and (38) do not depend on the starting point 0, we see that $E_0[v(\tau); A_3]$ is bounded above by $\lceil \tau/2 \rceil$ times the sum of the right hand side in (35) and (38), respectively. So, we have the same upper bound on $\mathbf{E}_\epsilon \otimes E_0[v(\tau); E \cap A_3]$ as on $A_1 \cup A_2$. The proof of Proposition 6 is now complete. \square

3.4 Proof of the Upper Estimates

Now we are ready to prove Theorem 1 and the upper estimates of Theorems 2 and 3. To this end, we are going to give an upper bound on the probability that (scaled) surviving process leaves $B^{(l)}(\omega_1, \omega_2)$ before time τ . We first note that for $(\omega_1, \omega_2) \in \Omega_1^c \cap E$ the starting point of the process is not contained in any B_l and thus $T_{B_l} = 0$. Therefore, no matter which B_l we pick for $B^{(l)}$ on $\Omega_1^c \cap E$, we have

$$\begin{aligned} \{T_{B^{(l)}}\} & \leq \tau \subset \{T_{B^{(l)}} \leq \tau, T_{\mathcal{T}} > \tau\} \cup \{T_{\mathcal{T}} \leq \tau\} \\ & \subset \{T_{B^{(l)}} \leq \tau, T_{\mathcal{T}} > \tau, E\} \cup \{T_{\mathcal{T}} > \tau, E^c \cup \{T_{\mathcal{T}} \leq \tau\}. \end{aligned}$$

Consequently, we find for large t that

$$\begin{aligned} & Q_t^{\mu, \nu}(T_{t^{1/(d+2)}B^{(l)}(\omega_1, \omega_2)} \leq t) \\ & \leq \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); T_{B^{(l)}} \leq \tau, T_{\mathcal{T}} > \tau, E] \\ & \quad + \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); E^c] \\ & \quad + \frac{1}{S_t^{\mu, \nu}} \mathbf{E}_\epsilon \otimes E_0[v(\tau); T_{\mathcal{T}} \leq \tau] \\ & \leq \exp\left\{-c_{14}\left(lt^{\frac{\alpha_3}{d+2}} \wedge t^{\frac{d-\alpha_1}{d+2}}\right)\right\} \end{aligned} \tag{39}$$

with $c_{14} = c_7 \wedge (1 + \gamma)$ using (27), Propositions 4 and 6.

Proof of Theorem 1 We set $l = \epsilon^{\alpha_2}$ as previously stated at the end of Sect. 3.3. Then we have

$$l t^{\frac{\alpha_3}{d+2}} \wedge t^{\frac{d-\alpha_1}{d+2}} = t^{\frac{\alpha_3-\alpha_2}{d+2}} \wedge t^{\frac{d-\alpha_1}{d+2}}$$

and therefore the right hand side of (39) converges to 0 as $t \rightarrow \infty$, which proves Theorem 1. □

Proof of the upper estimate of (5) Since on $\{T_{t^{1/(d+2)}B^{(l)}(\omega_1, \omega_2)} > t\}$ we have

$$W_t^C \subset a\text{-neighborhood of } t^{\frac{1}{d+2}} B^{(l)}(\omega_1, \omega_2) \tag{40}$$

and the volume of the right hand side of (40) is smaller than

$$t^{\frac{d}{d+2}} \omega_d (R_0 + \kappa_1 t^{-\frac{\kappa_2}{d+2}} + at^{-\frac{1}{d+2}})^d \sim t^{\frac{d}{d+2}} \omega_d R_0^d \quad (t \rightarrow \infty),$$

the upper estimate of Theorem 2 follows. □

Proof of Theorem 3 It is enough to show that for arbitrary $\eta > 0$ and $l > 2R_0$,

$$Q_t^{\mu, \nu} (T_{t^{1/(d+2)}B(0, l)} \leq t) \leq \exp\{-\eta l\} \tag{41}$$

when t is large enough. First of all, we can find a constant $M > 0$ such that for all $l > Mt^{d/(d+2)}$,

$$P_0(T_{t^{1/(d+2)}B(0, l)} \leq t) \leq \exp\{-\eta l - 2c(d, \mu + \nu)t^{\frac{d}{d+2}}\}$$

using a standard Brownian estimate. From this and (9), we have (41) in this case. Next, for $2R_0 < l \leq Mt^{d/(d+2)}$ we can use (39) to derive

$$\begin{aligned} Q_t^{\mu, \nu} (T_{t^{1/(d+2)}B(0, l)} \leq t) &\leq Q_t^{\mu, \nu} (T_{t^{1/(d+2)}B^{(l/2-R_0)}(\omega_1, \omega_2)} \leq t) \\ &\leq \exp\left\{-c_{14} \left(\left(\frac{l}{2} - R_0\right) t^{\frac{\alpha_3}{d+2}} \wedge \frac{l}{M^2} t^{\frac{1-\alpha_1}{d+2}}\right)\right\}. \end{aligned}$$

This implies (41) when t is large. In the remaining case $M^2 t^{d/(d+2)} < l \leq Mt^{d/(d+2)}$, we shall use large deviation estimate in [6] again to show

$$\limsup_{t \rightarrow \infty} t^{-\frac{d}{d+2}} \log Q_t^{\mu, \nu} \left(\sup_{0 \leq s \leq t} |Z_s| > xt^{\frac{d}{d+2}} \right) \leq - \inf_{y \notin B(0, x)} \beta_0(y). \tag{42}$$

Strictly speaking, [6] deals with large deviation estimates for $t^{-d/(d+2)} Z_t$ but the proof is obviously applicable to above version. The only property of β_0 we need here is that it is a norm. We can deduce from (42) that for large t ,

$$\begin{aligned} Q_t^{\mu, \nu} (T_{t^{1/(d+2)}B(0, l)} \leq t) &\leq Q_t^{\mu, \nu} \left(\sup_{0 \leq s \leq t} |Z_s| > M^2 t^{\frac{d}{d+2}} \right) \\ &\leq \exp\left\{-\frac{1}{2} M^2 t^{\frac{d}{d+2}} \inf_{y \notin B(0, 1)} \beta_0(y)\right\} \\ &\leq \exp\left\{-\frac{1}{2} Ml \inf_{y \notin B(0, 1)} \beta_0(y)\right\}. \end{aligned} \tag{43}$$

This implies (41) making M larger if necessary.

Now we have (41) for all $l > 2R_0$ and the proof of Theorem 3 is completed. □

Acknowledgements The author would like to thank Professor Nobuo Yoshida for suggesting this very interesting problem, a lot of helpful discussions, and careful reading of the early version of the manuscript. He is also grateful to the referee for many constructive suggestions.

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